Chapter 6.2.1

THE INVERSE SCATTERING PROBLEM AND ITS APPLICATIONS TO NLPDE

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PART 6 SCATTERING IN MATHEMATICS AND NONPHYSICAL SCIENCES

Topic 6.2 Inverse Scattering Transform and Non-linear Partial Differential Equations

Contents

§ 1. Scattering Problem for the 1D Schrödinger Operator

The one-dimensional **Schrödinger operator**

$$L = -\frac{d^2}{dx^2} + q(x), \quad -\infty < x < +\infty \tag{1}$$

with a real locally summable potential $q(x)$ satisfying the condition

$$\int_{-\infty}^{+\infty} (1+|x|)\,|q(x)|\,dx < \infty \tag{2}$$

is a self-adjoint operator and has a double absolutely continuous spectrum that fills the whole positive semiaxis and a finite number of simple negative eigenvalues.

The condition (2) implies that for any k from the closed upper (lower) half-plane, the equation

$$-y'' + q(x)y = k^2 y \tag{3}$$

has solutions $e^+(k,x)$ and $e^-(-k,x)(e^+(-k,x)$ and $e^-(k,x)$, respectively) that uniformly in k satisfy the asymptotic equalities

$$\begin{aligned}
e^+(k,x) &= e^{ikx}(1+o(1)), \quad x \to +\infty, \\
e^-(-k,x) &= e^{-ikx}(1+o(1)), \quad x \to -\infty
\end{aligned} \quad (\mathrm{Im}\,k \geq 0);$$
$$\begin{aligned}
e^+(-k,x) &= e^{-ikx}(1+o(1)), \quad x \to +\infty, \\
e^-(k,x) &= e^{ikx}(1+o(1)), \quad x \to -\infty
\end{aligned} \quad (\mathrm{Im}\,k \leq 0), \tag{4}$$

which remain valid after being once differentiated. For example,

$$\begin{aligned}
&\frac{d}{dx}e^+(k,x) = e^{ikx}(ik+o(1)), \quad x \to \infty \quad (\mathrm{Im}\,k \geq 0), \\
&\frac{d}{dx}e^-(k,x) = e^{ikx}(ik+o(1)), \quad x \to -\infty \quad (\mathrm{Im}\,k \leq 0),
\end{aligned} \tag{5}$$

and so on. They are called the **Jost solutions** and can be found from the integral equations

$$e^{\pm}(k,x) = e^{ikx} - \int_x^{\pm\infty} \frac{\sin k(x-t)}{k} q(t) e^{\pm}(k,x)\,dt \tag{6}$$

by iterations that converge uniformly in k in the corresponding half-planes. So, the solutions $e^+(k,x), e^-(-k,x)$ $(e^+(-k,x), e^-(k,x)$, respectively) are analytic in k in the half-plane $\mathrm{Im}\,k > 0$ $(\mathrm{Im}\,k < 0)$ and continuous in the closed half-plane $\mathrm{Im}\,k \geq 0$ $(\mathrm{Im}\,k \leq 0\,)$. For $|k| \to \infty$, they satisfy asymptotic equalities

$$\begin{aligned}
&e^{\pm}(\pm k,x) = e^{\pm ikx}\Big(1 \mp \tfrac{1}{2ik} \int_x^{\pm\infty} q(t)\,dt + o(k^{-1})\Big), \\
&\qquad |k| \to \infty\ (\mathrm{Im}\,k \geq 0), \\
&e^{\pm}(\mp k,x) = e^{\mp ikx}\Big(1 \pm \tfrac{1}{2ik} \int_x^{\pm\infty} q(t)\,dt + o(k^{-1})\Big), \\
&\qquad |k| \to \infty\ (\mathrm{Im}\,k \leq 0).
\end{aligned} \tag{7}$$

It follows from these properties and the Jordan lemma that for any potential $q(x)$ satisfying condition (2) there exist triangular transformation operators introduced by Levin (1956)

$$\begin{aligned}
e^+(\pm k,x) &= e^{\pm kx} + \int_x^{\infty} K^+(x,t)e^{\pm ikt}\,dt \quad (\pm\mathrm{Im}\,k \geq 0) \\
e^-(\pm k,x) &= e^{\pm kx} + \int_{-\infty}^{x} K^-(x,t)e^{\pm ikt}\,dt \quad (\pm\mathrm{Im}\,k \leq 0)
\end{aligned} \tag{8}$$

that transform the functions $e^{\pm ikx}$ (that is, Jost solutions of Eq. (3) with the zero potential) into Jost solutions of Eq. (3).

One can obtain (Agranovich and Marchenko, 1960) the properties of kernels $K^{\pm}(x,t)$ of these operators from integral equations connecting them directly to the potential $q(x)$. With this purpose one must substitute the right-hand sides of equalities (8) into Eq. (6) and then make a Fourier transform with respect to k. As a result one gets the equations

$$\begin{aligned}
K^+(x,t) = \frac{1}{2}\int_{\frac{x+t}{2}}^{\infty} q(\xi)\,d\xi - \int_{\frac{x+t}{2}}^{\infty} d\eta \int_0^{\frac{t-x}{2}} q(\eta-\xi) \\
\times K^+(\eta-\xi,\eta+\xi)\,d\xi,
\end{aligned}$$

$$K^{-}(x,t) = \frac{1}{2}\int_{-\infty}^{\frac{x+t}{2}} q(\xi)d\xi - \int_{-\infty}^{\frac{x+t}{2}} d\eta \int_{\frac{t-x}{2}}^{0} q(\eta-\xi) \times K^{-}(\eta-\xi,\eta+\xi)d\xi.$$

After solving them by iteration, one finds their direct corollaries, which are equalities

$$K^{+}(x,x) = \frac{1}{2}\int_{x}^{\infty} q(t)dt, \quad K^{-}(x,x) = \frac{1}{2}\int_{-\infty}^{x} q(t)dt, \tag{9}$$

and the estimates

$$\left| K^{\pm}(x,t) \mp \frac{1}{2}\int_{\frac{x+t}{2}}^{\pm\infty} q(\xi)d\xi \right| \leq \frac{1}{2}\sigma^{\pm}\left(\frac{x+t}{2}\right)\exp\sigma_1^{\pm}(x),$$

$$\left| \frac{\partial K^{\pm}(x_1,x_2)}{\partial x_i} \pm \frac{1}{4}q\left(\frac{x_1+x_2}{2}\right)\right| \leq \frac{1}{2}\sigma^{\pm}\left(\frac{x_1+x_2}{2}\right)\sigma_1^{\pm}(x_1) \times\exp\sigma_1^{\pm}(x_1) \tag{9'}$$

where $i = 1,2$, and

$$\sigma^{\pm}(x) = \pm\int_{x}^{\pm\infty} |q(\xi)|d\xi, \; \sigma_1^{\pm}(x) = \pm\int_{x}^{\pm\infty} \sigma^{\pm}(\xi)d\xi.$$

If the potential is differentiable and its derivative satisfies condition (2), then the kernels $K^{\pm}(x,t)$ are twice differentiable and satisfy the equation

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right)K^{\pm}(x,t) - q(x)K^{\pm}(x,t) = 0. \tag{10}$$

In this case the functions $K^{\pm}(x,t)$ are the Levin transformation operator kernels if and only if they satisfy Eq. (10), equalities (9), and the limit conditions

$$\lim_{x_1+x_2\to\pm\infty} \frac{\partial K^{\pm}(x_i,x_2)}{\partial x_i} \pm \frac{1}{4}q\left(\frac{x_1+x_2}{2}\right) = 0$$

in the area $\pm(x_1 - x_2) > 0$.

For real k all the four Jost solutions are well defined, and it follows from the asymptotic formulae (4) that solutions $e^{+}(k,x)$, $e^{+}(-k,x)$ (as well as $e^{-}(k,x)$, $e^{-}(-k,x)$) are linearly independent for $k \neq 0$. Therefore, for real $k \neq 0$, any solution $y(k,x)$ of (3) is their linear combination:

$$y(k,x) = C_1^{\pm}e^{\pm}(k,x) + C_2^{\pm}e^{\pm}(-k,x).$$

The coefficients may be found from the equalities

$$W\{y(k,x), e^{\pm}(-k,x)\} = C_1^{\pm}(k)W\{e^{\pm}(k,x), e^{\pm}(-k,x)\},$$
$$W\{y(k,x), e^{\pm}(k,x)\} = -C_2^{\pm}(k)W\{e^{\pm}(k,x), e^{\pm}(-k,x)\},$$

where $W\{f,g\} = f'(x)g(x) - f(x)g'(x)$ stands for the Wronskian of f and g. Since the Wronskian of the solutions of (3) is independent of x, it follows from the asymptotic formulae (4), (5) that

$$W\{e^{\pm}(k,x), e^{\pm}(-k,x)\} = \lim_{x\to\pm\infty} W\{e^{\pm}(k,x), e^{\pm}(-k,x)\} = 2ik. \tag{11}$$

In particular,

$$e^{+}(k,x) = a(k)e^{-}(k,x) + b(k)e^{-}(-k,x),$$
$$e^{-}(-k,x) = a(k)e^{+}(-k,x) - b(-k)e^{+}(k,x), \tag{12}$$

with

$$a(k) = \frac{W\{e^{+}(k,x), e^{-}(-k,x)\}}{2ik},$$
$$b(k) = \frac{W\{e^{-}(k,x), e^{+}(k,x)\}}{2ik}. \tag{13}$$

Substituting the right-hand sides of (12) into the formula (11) instead of $e^{+}(\pm k,x)$, we find that

$$a(k)a(-k) - b(k)b(-k) = 1,$$

and since $e^{\pm}(k,x) = \overline{e^{\pm}(-k,x)}$, we get $a(k) = \overline{a(-k)}$, $b(k) = \overline{b(-k)}$ and

$$|a(k)|^2 - |b(k)|^2 = 1. \tag{14}$$

This implies in particular that $|a(k)|^2 \geq 1$. The last inequality makes it possible to divide (12) by $a(k)$ and introduce the solutions

$$u^{-}(k,x) = e^{-}(k,x) + r^{-}(k)e^{-}(-k,x) = d(k)e^{+}(k,x),$$
$$u^{+}(k,x) = e^{+}(-k,x) + r^{+}(k)e^{+}(k,x) = d(k)e^{-}(-k,x),$$

where

$$d(k) = a(k)^{-1}, \; r^{-}(k) = \frac{b(k)}{a(k)},$$
$$r^{+}(k) = -\frac{b(-k)}{a(k)} = -\frac{a(-k)}{a(k)}r^{-}(-k) \tag{15}$$

and, due to (14),

$$|d(k)|^2 + |r^{\pm}(k)|^2 = 1. \tag{16}$$

The functions $u^{-}(k,x)$ and $u^{+}(k,x)$ form the complete collection of generalised eigenfunctions of the continuous spectrum of the operator L and they satisfy the asymptotic equalities

$$u^{-}(k,x) \sim e^{ikx} + r^{-}(k)e^{-ikx}, \quad x \to -\infty$$
$$u^{-}(k,x) \sim d(k)e^{ikx}, \quad x \to \infty$$
$$u^{+}(k,x) \sim e^{-ikx} + r^{+}(k)e^{ikx}, \quad x \to \infty$$
$$u^{+}(k,x) \sim d(k)e^{-ikx}, \quad x \to -\infty.$$

A free particle moving in the positive (negative) direction of the x axis with momentum k is described in nonrelativistic quantum mechanics by the plane wave $e^{ikx}(e^{-ikx})$ with $k > 0$. The solution $u^{-}(k,x)$ $(u^{+}(k,x))$ describes the scattering of particles by the potential $q(x)$. The function $d(k)e^{ikx}$ $(d(k)e^{ikx})$ corresponds to particles that have come to $+\infty$ $(-\infty)$, while $r^{-}(k)e^{ikx}$ $(r^{+}(k)e^{ikx})$ to those that are reflected to the left-(right-) hand side.

In correspondence with this interpretation, the function $u^{-}(k,x)$ $(u^{+}(k,x))$ is called the eigenfunction of the left (right) scattering problem and the coefficients $r^{-}(k)$ $(r^{+}(k))$ and $d(k)$ the left (right) **reflection coefficient** and the transmission coefficient, respectively.

For complex k from the upper half-plane, $e^{+}(k,x)$ $(e^{-}(-k,x))$ is the only linear independent solution of Eq. (3) belonging to the space $L^{2}(0,\infty)$ $(L^{2}(-\infty,0))$. Hence this equation has a solution belonging to the space $L^{2}(-\infty,\infty)$ only for such values $k(\operatorname{Im} k>0)$ that make solutions $e^{+}(k,x)$ and $e^{-}(k,x)$ linearly dependent, that is, when

$$W\{e^{+}(k,x),e^{-}(-k,x)\}=2ik\cdot a(k)=0.$$

Then the solutions $u^{-}(k_l,x)=e^{-}(-k_l,x)$ and $u^{+}(k_l,x)=e^{+}(k_l,x)$ are linearly dependent,

$$u^{-}(k_l,x)=b_l^{-}u^{+}(k_l,x);\quad u^{+}(k_l,x)=b_l^{+}u^{-}(k_l,x),\quad (17)$$

$b_l^{+}b_l^{-}=1$, and belong to the space $L^{2}(-\infty,\infty)$ if and only if $a(k_l)=0$. They are called the left $(u^{-}(k_l,x))$ and the right $(u^{+}(k_l,x))$ eigenfunctions of the scattering problem corresponding to the eigenvalue k_l^{2}. Since the operator L is self-adjoint, its eigenvalues are real. Hence $k_l=i\kappa_l$ $(\kappa_l>0)$, so zeros of the function $a(k)$ may be situated on the imaginary semiaxis $i\kappa$ $(\kappa>0)$, while the condition (2) guarantees that there are only a finite number of them. The inverse values of the squares of the $L^{2}(-\infty,\infty)$ norms of the eigenfunctions of the discrete spectrum are called the normalising coefficients,

$$m_l^{\pm}=\|u^{\pm}(k_l,x)\|^{-2},$$

while the collections $\{r^{-}(k),i\kappa_l,m_l^{-}\}$ and $\{r^{+}(k),i\kappa_l,m_l^{+}\}$ are called the left and the right **scattering data**, respectively. The **inverse scattering problem** is to reconstruct the potential $q(x)$ from the left or the right scattering data and to find the necessary and sufficient conditions that an arbitrary collection $\{r(k),i\kappa_l,m_l\}$ must satisfy in order to be the left or the right scattering data for some Schrödinger operator with the real potential $q(x)$, satisfying condition (2). One can express the normalising coefficients via the coefficients $b_l^{\pm}$ and the derivatives $\dot a(k_l)=\frac{\partial a(k)}{\partial k}|_{k=k_l}$ by the equalities

$$\frac{\partial}{\partial x}W\{e^{\pm}(\pm k,x),\dot e^{\pm}(\pm k,x)\}=2ke^{\pm}(\pm k,x)^{2}.$$

To prove it note that due to (17) we have

$$\begin{aligned}
(m_l^{\pm})^{-1}&=\|u^{\pm}(k_l,x)\|^{2}\\
&=b_l^{\pm}\left\{b_l^{+}\int_{-\infty}^{0}e^{-}(-k_l,x)^{2}dx+b_l^{-}\int_{0}^{\infty}e^{+}(k_l,x)^{2}dx\right\}\\
&=\frac{b_l^{\pm}}{2k_l}\big[b_l^{+}W\{e^{-}(-k_l,x),\dot e^{-}(-k_l,x)\}\\
&\quad-b_l^{-}W\{e^{+}(k_l,x),\dot e^{+}(k_l,x)\}\big]|_{x=0}\\
&=\frac{b_l^{\pm}}{2k_l}[W\{e^{+}(k_l,x),\dot e^{-}(-k_l,x)\}\\
&\quad+W\{\dot e^{+}(k_l,x),e^{-}(-k_l,x)\}]|_{x=0}\\
&=\frac{b_l^{\pm}}{2k_l}\frac{\partial}{\partial k}2ika(k)|_{k=k_l}=\frac{b_l^{\pm}}{2k_l}\{2ia(k_l)+2ik_l\dot a(k_l)\}
\end{aligned}$$

and since $a(k_l)=0$, we get

$$(m_l^{\pm})^{-1}=ib_l^{\pm}\dot a(k_l)\quad(k_l=i\kappa_l,\ \kappa_l>0).\qquad(18)$$

In particular, it follows from this equality that all the zeros $k_l=i\kappa_l$ of the function $a(k)$ are simple. It follows from the estimates (9'), and formulae (8) and (13) that

$$2ika(k)=2ik+\int_{-\infty}^{\infty}q(x)dx+\int_{-\infty}^{\infty}A(x)e^{-ikx}dx,$$

$$2ikb(k)=\int_{-\infty}^{\infty}B(x)e^{-ikx}dx,$$

where $A(x),B(x)\in L^{1}(-\infty,\infty)$ and $A(x)=0$ for $x>0$. Therefore, the reflection coefficients and their Fourier transforms

$$R^{\pm}(y)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}r^{\pm}(k)e^{\pm iky}dk$$

belong to $L^{2}(-\infty,\infty)$.

The main integral equations of the scattering theory, which allow the reconstruction of the potential $q(x)$ from the given left or right scattering data, can be obtained by applying the Fourier transform to equalities (12) (see Kay and Moses (1956) and Faddeev (1964)). After rewriting (12) as

$$\begin{aligned}
\left(\frac{1}{a(k)}-1\right)e^{-}(-k,x)&=r^{+}(k)e^{+}(k,x)+e^{+}(-k,x)-e^{-}(-k,x)\\
&=r^{+}(k)\{e^{ikx}+\int_{-\infty}^{+\infty}K^{+}(x,t)e^{ikt}dt\}\\
&\quad+\int_{-\infty}^{+\infty}\{K^{+}(x,t)-K^{-}(x,t)\}e^{-ikt}dt,
\end{aligned}$$

we find that the Fourier transform of the right-hand side is equal to

$$R^{+}(x+y)+K^{+}(x,y)+\int_{x}^{+\infty}R^{+}(x+t)K^{+}(x,t)dt-K^{-}(x,y),\ (19)$$

while the Fourier transform of the left-hand side can be calculated for $y>x$ via the residue theory:

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\frac{1}{a(k)}-1\right)e^{-}(-k,x)e^{iky}dk=i\sum_{l}\frac{e^{-}(-i\kappa_l,x)e^{-\kappa_l y}}{\dot a(ik_l)}.$$

Using (8), (17) and (18), we may reduce this to

$$-\sum_{l}m_l^{+}\left\{e^{-\kappa_l(x+y)}+\int_{x}^{\infty}K^{+}(x,t)e^{-\kappa_l(x+y)}\right\}.\qquad(20)$$

Therefore, the functions (19) and (20) coincide for $y>x$. Since $K^{-}(x,y)=0$ for $y>x$, it follows that

$$F^{+}(x+y)+K^{+}(x,y)+\int\limits_{x}^{+\infty} F^{+}(x+t)K^{+}(x,t)dt=0 \quad (y>x), \tag{21}$$

where the function

$$F^{+}(z)=\sum_{l} m_{l}^{+}e^{-\kappa_{l}z}+R^{+}(z)=\sum_{l} m_{l}^{+}e^{-\kappa_{l}z} + \frac{1}{2\pi}\int_{-\infty}^{+\infty} r^{+}(k)e^{ikz}dk \tag{21'}$$

is defined by the right scattering data. The same considerations yield for $y<x$ the equality

$$F^{-}(x+y)+K^{-}(x,y)+\int\limits_{-\infty}^{x} F^{-}(y+t)K^{-}(x,t)dt=0 \quad (y<x) \tag{22}$$

with the function

$$F^{-}(z)=\sum_{l} m_{l}^{-}e^{\kappa_{l}z}+\frac{1}{2\pi}\int_{-\infty}^{+\infty} r^{-}(k)e^{-ikz}dk \tag{22'}$$

defined by the left scattering data. For any fixed x equalities (21) and (22) play the role of main integral equations for kernels of the transformation operators, $K^{+}(x,y)$ when $x<y<\infty$ and $K^{-}(x,y)$ when $-\infty<y<x$, considered as functions of the variable y. The same equalities together with estimates (9') show that both functions $F^{\pm}(x)$ are absolutely continuous and satisfy the inequalities

$$\begin{aligned}\left|F^{\pm}(2x)\pm\tfrac{1}{2}\int\limits_{x}^{\pm\infty} q(t)dt\right| &\leq \tfrac{\sigma^{\pm}(x)}{4}e^{2\sigma_{1}^{\pm}(x)},\\ \left|\tfrac{d}{dx}F^{\pm}(2x)\mp\tfrac{1}{2}q(x)\right| &\leq (\sigma^{\pm}(x))^{2}e^{2\sigma_{1}^{\pm}(x)},\end{aligned} \tag{23}$$

which implies in particular that for any $a>-\infty$,

$$\int\limits_{a}^{\infty} |F^{\pm}(\pm x)|\,dx<\infty; \qquad \int\limits_{a}^{\infty} (1+|x|)\left|\frac{d}{dx}F^{\pm}(\pm x)\right|dx<\infty.$$

If we introduce the variables $\eta=y-x, \tau=t-x$, the main equations will take the form

$$F^{\pm}(2x+\eta)+K^{\pm}(x,x+\eta)\pm\int\limits_{0}^{\pm\infty} F^{\pm}(2x+\eta+\tau)K^{\pm}(x,x+\tau)d\tau=0,$$

with integral operators $\mathbf{F}^{\pm}$ with kernels $F^{\pm}(2x+\eta+\tau)$ depending on the parameter x acting in the fixed spaces $L^{2}(0,\infty)$ and $L^{2}(-\infty,0)$, respectively. It follows from the estimate (23) that they are self-adjoint Hilbert–Schmidt-type operators. Therefore, in order to prove the existence of a solution of the main integral equations it suffices to check that the corresponding homogeneous equations $(\mathbf{I}+\mathbf{F}^{\pm})f=0$ have no solutions distinct from 0. Direct computations (using the Parceval's equalities for the ususal Fourier transforms) leads us to inequalities

$$((\mathbf{I}+\mathbf{F}^{\pm})f,f)\geq\frac{1}{2\pi}\int_{-\infty}^{\infty}(1-|r^{+}(k)|)^{2}\left|\tilde{f}(k)\right|^{2}dk,$$
$$\tilde{f}(k)=\int_{0}^{\infty} f(t)e^{-ikt}dt,$$

where $f(t)\in L^{2}(0,\infty)$ and $(\cdot,\cdot)$ denotes the scalar product in $L^{2}(0,\infty)$. Since $|r^{+}(k)|<1$ for all $k\neq 0$, the right-hand side of the inequality may be equal to 0 only if $\tilde{f}(k)\equiv 0$, so the equation $(\mathbf{I}+\mathbf{F}^{+})f=0$ has only the zero solution, and the operator $(\mathbf{I}+\mathbf{F}^{+})$ is invertible. The invertibility of the operator $(\mathbf{I}+\mathbf{F}^{-})$ can be proved the same way. It is therefore proved that the main integral equations have unique solutions. After solving them (for each value of the parameter $x\in(-\infty,\infty)$), one gets kernels of the transformation operators $K^{\pm}(x,y)$ that reconstruct the potential using formulae (9).

The kernels $F^{\pm}(2x+\eta+\xi)$ of operators $\mathbf{F}^{\pm}$ satisfy the obvious equalities

$$\frac{\partial}{\partial x}F^{\pm}=2\frac{\partial}{\partial\eta}F^{\pm}=2\frac{\partial}{\partial\xi}F^{\pm},$$

which allows us to express the potential $q(x)$ via the Fredholm determinant of the operator $\mathbf{I}+\mathbf{F}^{+}$ (or $\mathbf{I}+\mathbf{F}^{-}$). Actually, if λ_{k} and $\omega_{k}(\eta)$ are the eigenvalues and orthonormal eigenfunctions of the operator $\mathbf{F}^{+}$, then

$$F^{+}(2x+\eta+\xi)=\sum_{k=1}^{\infty}\lambda_{k}\omega_{k}(\eta)\omega_{k}(\eta), \tag{24}$$

and it follows from here and (21) that

$$\begin{aligned}K^{+}(x,x+\eta)&=-(\mathbf{I}+\mathbf{F}^{+})^{-1}F^{+}(2x+\eta+0)\\&=-(\mathbf{I}+\mathbf{F}^{+})^{-1}\left(\sum_{k=0}^{\infty}\lambda_{k}\omega_{k}(\eta)\omega_{k}(0)\right)\\&=-\sum_{k=0}^{\infty}\frac{\lambda_{k}}{1+\lambda_{k}}\omega_{k}(\eta)\omega_{k}(0))\end{aligned}$$

and

$$K^{+}(x,x)=-\sum_{k=1}^{\infty}\frac{\lambda_{k}}{1+\lambda_{k}}\omega_{k}(0)^{2}. \tag{25}$$

On the other hand, we have

$$\frac{d}{dx}\mathrm{Sp}(\mathbf{F}^{+})^{p}=\mathrm{Sp}\frac{d}{dx}(F^{+})^{p}=\sum_{j=0}^{p-1}\mathrm{Sp}\left\{(\mathbf{F}^{+})^{j}\left(\frac{d}{dx}(\mathbf{F}^{+})\right)(\mathbf{F}^{+})^{p-j-1}\right\},$$

and since (24) implies that the kernels of the operators $(\mathbf{F}^{+})^{p}$ and $d(\mathbf{F}^{+})/dx$ are equal to

$$F_{p}^{+}(\eta,\xi)=\sum_{k=1}^{\infty}\lambda_{k}^{p}\omega_{k}(\eta)\omega_{k}(\xi),$$

and

$$\begin{aligned}\frac{\partial}{\partial x}F^{+}(2x+\eta+\xi)&=2\frac{\partial}{\partial\eta}F^{+}(2x+\eta+\xi)\\&=2\sum_{k=1}^{\infty}\lambda_{k}\omega_{k}(\xi)\frac{\partial}{\partial\eta}\omega_{k}(\eta),\end{aligned}$$

we get the formula

$$\frac{d}{dx}\left(\sum_{k=1}^{\infty}\lambda_k^p\right) = \frac{d}{dx}\mathrm{Sp}(\mathbf{F}^+)^p$$

$$= 2\sum_{j=0}^{p-1}\int_0^\infty d\eta\int_0^\infty\int_0^\infty F_j^+(\eta,\alpha)\frac{\partial}{\partial\alpha}$$

$$\times F^+(2x+\alpha+\beta)F_{p-j-1}^+(\beta,\eta)d\beta d\alpha$$

$$= 2\sum_{j=0}^{p-1}\int_0^\infty\int_0^\infty\frac{\partial}{\partial\alpha}F^+(2x+\alpha+\beta)$$

$$\times(\int_0^\infty F_j^+(\eta,\alpha)F_{p-j-1}^+(\eta,\beta)d\eta)d\beta d\alpha$$

$$= 2p\int_0^\infty\int_0^\infty\left(\sum_{k=1}^{\infty}\lambda_k\omega_k(\beta)\frac{\partial\omega_k(\alpha)}{\partial\alpha}\right)\cdot$$

$$\left(\sum_{s=1}^{\infty}\lambda_s^{p-1}\omega_s(\alpha)\omega_s(\beta)\right)d\beta d\alpha$$

$$= 2p\int_0^\infty\sum_{k=1}^{\infty}\lambda_k^p\omega_k(\alpha)\frac{\partial\omega_k(\alpha)}{\partial\alpha}d\alpha$$

$$= -p\sum_{k=1}^{\infty}\lambda_k^p\omega_k(0)^2.$$

Hence

$$\frac{d}{dx}\sum_{k=1}^{\infty}\frac{(z\lambda_k)^p}{p} = -\sum_{k=1}^{\infty}(z\lambda_k)^p\omega_k(0)^2,$$

and for $|z|$ sufficiently small

$$\frac{d}{dx}\sum_{k=1}^{\infty}\ln(1-z\lambda_k) = \frac{d}{dx}\sum_{k=1}^{\infty}\sum_{p=1}^{\infty}\frac{(z\lambda_k)^p}{p}$$

$$= -\sum_{k=1}^{\infty}\sum_{p=1}^{\infty}(z\lambda_k)^p\omega_k(0)^2$$

$$= -\sum_{k=1}^{\infty}\frac{z\lambda_k}{1-z\lambda_k}\omega_k(0)^2.$$

Since both sides of this equality are analytic in z for $z=-1$, it follows from it and (25) that

$$K^+(x,x) = \frac{d}{dx}\sum_{k=1}^{\infty}\ln(1+\lambda_k)$$

$$= \frac{d}{dx}\ln\prod_{k=1}^{\infty}(1+\lambda_k)$$

$$= \frac{d}{dx}\ln\,\mathrm{Det}(\mathbf{I}+\mathbf{F}^+)$$

and

$$q(x) = -2\frac{d}{dx}K^+(x,x) = -2\frac{d^2}{dx^2}\ln\,\mathrm{Det}(\mathbf{I}+\mathbf{F}^+).$$

The formula

$$q(x) = -2\frac{d^2}{dx^2}\ln\,\mathrm{Det}(\mathbf{I}+\mathbf{F}^-).$$

can be proved in the same way.

If the reflection coefficients $r^\pm(k)$ are rational fractions, then the kernels $F^\pm(2x+\eta+\xi)$ are degenerate, and the Fredholm determinants are of finite order whose elements can be written explicitly. The potential can be explicitly expressed in terms of the scattering data in this case. For example, if $r^\pm(k)\equiv 0$, then

$$q(x) = -2\frac{d^2}{dx^2}\ln\mathrm{Det}\,\Delta^\pm, \tag{26}$$

where $\Delta^\pm$ is a matrix of the nth order with the elements

$$\Delta_{kl}^\pm = \delta_{kl} + \frac{m_k^\pm e^{\mp(\kappa_k+\kappa_l)x}}{\kappa_k+\kappa_l},$$

and δ_{kl} is the Kronecker symbol. Such potentials are called reflectionless or Bargman potentials.

The characteristic properties of the scattering data that are necessary and sufficient conditions under which an arbitrary set

$$\{r(k),\kappa_p,m_p\}(-\infty<k<\infty,\quad p=1,2,\ldots,n) \tag{27}$$

is right or left scattering data for some operator (1) with real potential $q(x)$ depend on the class of potential under consideration. For potentials from the Schwartz space (that is, infinitely differentiable and decreasing as $|x|\to\infty$ faster than any power $|x|^{-m}$ together with all derivatives) besides the obvious conditions

(I) $r(k)=\overline{r(-k)}, |r(k)|<1$ for $k \neq 0, \kappa_p>0, m_p>0$; it is necessary and sufficient that the function $r(k)$ belongs to the Schwartz space and satisfy the inequalities

(III) $-1\le r(0)<1$. (In the case $r(0)=-1$, an extra condition

$$0<\lim_{k\to 0}\frac{1-|r(k)|}{k^2}<\infty$$

must hold.)

The scattering data of the widest class of potentials that satisfy only inequality (2) besides conditions (I) and (III) satisfy also the following conditions:

(II) The function

$$R(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}r(k)e^{ikx}dx$$

is absolutely continuous and belongs to the space $L^1(-\infty,\infty)$ together with its derivative;

(IV) if $r(0)=-1$, then the function

$$M(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1-|r(k)|^2}{k^2}e^{ikx}dk$$

is absolutely continuous and belongs to the space $L^1(\infty,-\infty)$ together with its derivative.

These properties, the formula (16) and the analyticity of the coefficient $a(k)$ in the upper half-plane imply the following formula for the transmission coefficient $d(k)=a(k)^{-1}$